Journal of Algebra **232**, 541–563 (2000)
doi:10.1006/jabr.2000.8400, available online at http://www.idealibrary.com on IDEAL®

Tilting Equivalences for Grothendieck Categories

Enrico Gregorio

Dipartimento Scientifico e Tecnologico, Università di Verona, Strada le Grazie

Communicated by Kent R. Fuller

Received May 4, 1999

INTRODUCTION

In the past few years the study of tilting modules over arbitrary rings has been a very active subject. Many authors have succeeded in establishing and then generalizing theorems in this field—among them Happel and Ringel, Assem, Bongartz, Miyashita, Colby and Fuller. One of their goals was to state a general form of the celebrated Brenner and Butler theorem [4] and to extend results about tilting torsion theories (i.e., torsion theories generated by tilting modules) and tilting counter-equivalences. Recently Colpi [7] characterized tilting modules in terms of classes of modules and later [8] he was able to establish a theory of tilting objects in a Grothendieck category; he showed, in particular, that a tilting module in a closed category of modules (see Remark 1.2) is just a $*$-module.

During the meeting "Some trends in algebra" in Prague (1997), we had a conversation with R. Colpi and H. Krause regarding the possibility of studying tilting counter-equivalences (à la Colby and Fuller [6]) between two Grothendieck categories. This paper is a step in this direction.

First (Sections 1 and 3) we study the general situation, giving generalizations of results about tilting counter-equivalences. Let $(\mathscr{T}_1, \mathscr{F}_1)$ and $(\mathscr{T}_2, \mathscr{F}_2)$ be torsion theories in the Grothendieck categories $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, such that (1) every object in $\mathscr{C}_1$ is a subobject of an object in $\mathscr{T}_1$ and (2) every object in $\mathscr{C}_2$ is a quotient object of an object in $\mathscr{F}_2$. Assume we are given a pair of adjoint functors $F\colon \mathscr{C}_1 \to \mathscr{C}_2$, $G\colon \mathscr{C}_2 \to \mathscr{C}_1$ (F right adjoint to G) such that F and G induce an equivalence between $\mathscr{T}_1$ and $\mathscr{F}_2$: this setting will be named $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence. We prove that $\mathscr{T}_1$ is the kernel of the first right derived functor of F and the other right

0021-8693/00 $35.00

derived functors of F are zero. We show also that, even when the categories involved here have not enough projectives, it is possible to define a left derived functor of G in such a way that $\mathscr{F}_2$ is the kernel of this derived functor. Thus we are able to give a wide generalization of the Brenner and Butler theorem (see Theorem 3.7). In particular we find that the derived functors of F and G induce an equivalence between $\mathscr{F}_1$ and $\mathscr{T}_2$.

In Section 2 we show another approach to tilting equivalences, starting with a pair of adjoint functors; this approach shows that "maximal equivalences" are indeed tilting equivalences, provided we restrict in a suitable way the categories involved.

In Section 4 we apply the theory to the case of rings with local units, getting some generalizations of results by Colpi [8].

Section 5 analyzes the behavior of finitely generated objects under tilting equivalences and an open problem is presented.

Notations and conventions. All categories and functors will be additive. Every subcategory we consider is full and is assumed to contain every object isomorphic to one of its objects. If $F\colon \mathscr{A} \to \mathscr{B}$ is a functor, we denote by $\operatorname{im} F$ the full subcategory of $\mathscr{B}$ consisting of the objects isomorphic to $F(X)$, for $X \in \mathscr{A}$, and by $\ker F$ the full subcategory of $\mathscr{A}$ consisting of the objects $X \in \mathscr{A}$ such that $F(X) \cong 0$.

Our notion of *torsion theory* in a Grothendieck category is the usual one: in particular we do *not* require that torsion theories are hereditary.

Direct limits will be called *colimits*. Rings will be unital, with the exception of Section 4, where non-unital rings are considered. All modules will be unital, where a right R-module M is unital if $MR = M$.

I have to thank the "Cotilting" group: Lidia Angeleri Hügel, Gabriella D'Este, Alberto Tonolo, Jan Trlifaj, and especially Riccardo Colpi for many fruitful conversations about this subject.

I also have to thank the referee for many suggestions on how to improve the exposition.

1. TILTING EQUIVALENCES BETWEEN GROTHENDIECK CATEGORIES

1.1. Definition. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Grothendieck categories. A $\mathscr{C}_1$-$\mathscr{C}_2$-*pretilting equivalence* consists of:

(a) a torsion theory $(\mathscr{T}_1, \mathscr{F}_1)$ on $\mathscr{C}_1$;

(b) a torsion theory $(\mathscr{T}_2, \mathscr{F}_2)$ on $\mathscr{C}_2$;

(c) an equivalence $F\colon \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 :G$.

A $\mathscr{C}_1$-$\mathscr{C}_2$-pretilting equivalence is called a $\mathscr{C}_1$-$\mathscr{C}_2$-*tilting equivalence* if:

(1) every object of $\mathscr{C}_1$ is isomorphic to a subobject of an object in $\mathscr{T}_1$;

(2) every object of $\mathscr{C}_2$ is isomorphic to a quotient of an object in $\mathscr{F}_2$.

Note that condition (1) is equivalent to the requirement that every injective object in $\mathscr{C}_1$ belongs to $\mathscr{T}_1$; condition (2) implies that every projective object in $\mathscr{C}_2$ belongs to $\mathscr{F}_2$.

Since either the torsion class or the torsion-free class identifies a torsion theory, we can speak of the $\mathscr{C}_1$-$\mathscr{C}_2$-pretilting equivalence $F: \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 : G$.

The generalizations of the Brenner–Butler theorem given by Colby and Fuller tell us that every tilting module P_R (where R is a ring) yields a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence, where $\mathscr{C}_1 = \text{Mod-}R$ and $\mathscr{C}_2 = \text{Mod-End}(P_R)$, $\mathscr{T}_1 = \text{Gen}(P_R)$, $F = \text{Hom}_R(P, \cdot)$, $\mathscr{F}_2 = \text{im}\, F$ and $G = \cdot \otimes_{\text{End}(P_R)} P$.

1.2. *Remark*. A very important class of Grothendieck categories is provided by *closed categories of modules*: let R be a ring and consider a subcategory $\mathscr{C}$ of Mod-R closed under submodules, quotients, and direct sums: then $\mathscr{C}$ is a Grothendieck category. In the paper [8], Colpi showed that a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence, where $\mathscr{C}_1$ is a closed category of modules and $\mathscr{C}_2 = \text{Mod-}S$, for some ring S, is represented by a $*$-module P, in the sense that $S \cong \text{End}(P_R)$, $F \cong \text{Hom}_R(P, \cdot)$, and $G \cong \cdot \otimes_S P$.

Closed categories of modules in Mod-R are in one-to-one correspondence with right linear topologies on R. This class of Grothendieck categories has been extensively studied—see, for example, Wisbauer's book [26].

1.3. PROPOSITION. *Assume a $\mathscr{C}_1$-$\mathscr{C}_2$-pretilting equivalence $F: \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 : G$ is given. Then there exist functors $\tilde{F}: \mathscr{C}_1 \to \mathscr{C}_2$ and $\tilde{G}: \mathscr{C}_2 \to \mathscr{C}_1$ which extend F and G; moreover $\tilde{F}$ can be chosen as a right adjoint to $\tilde{G}$.*

Proof. Let t_i be the torsion radical associated to the torsion theory $(\mathscr{T}_i, \mathscr{F}_i)$ $(i = 1, 2)$. Define an action on the objects as

$$\tilde{F}(M) = F(t_1(M)) \qquad \text{and} \qquad \tilde{G}(N) = G(N/t_2(N)).$$

The action on morphisms is the obvious one. It is only a matter of calculations to show that indeed $\tilde{F}$ is a right adjoint of $\tilde{G}$. ∎

When we will be given a pretilting equivalence, we will always consider the functors extended as in the preceding proposition and write simply F and G, instead of $\tilde{F}$ and $\tilde{G}$. In view of this, the definition of a pretilting equivalence can be reformulated as follows: a $\mathscr{C}_1$-$\mathscr{C}_2$-*pretilting equivalence* consists of:

(1) a torsion theory $(\mathcal{T}_1, \mathcal{F}_1)$ on $\mathcal{C}_1$;

(2) a torsion theory $(\mathcal{T}_2, \mathcal{F}_2)$ on $\mathcal{C}_2$;

(3) a pair of functors $F: \mathcal{C}_1 \rightleftarrows \mathcal{C}_2 :G$, such that F is a right adjoint of G, $\operatorname{im} G = \mathcal{T}_2$, and $\operatorname{im} F = \mathcal{F}_2$;

(4) an equivalence induced by F and G between their images.

The unit and the counit of the adjunction will be denoted respectively by

$$\sigma_N : N \to FG(N) \qquad \text{and} \qquad \rho_M : GF(M) \to M.$$

Note that F and G induce an equivalence between their images if and only if the unit and the counit are isomorphisms for objects in the images [18]. Easy arguments yield that σ_N is surjective for all $N \in \mathcal{C}_2$ and ρ_M is injective for all $M \in \mathcal{C}_1$ (see Section 2).

A useful lemma was discovered by Colpi and Menini [9, Proposition 1.1] and generalized by Colpi [8, Lemma 1.5].

1.4. LEMMA. *Let $\mathcal{G}$ and $\mathcal{H}$ be Abelian categories; let $\mathcal{G}_0 \subseteq \mathcal{G}$ and $\mathcal{H}_0 \subseteq \mathcal{H}$ be full subcategories, each one closed either under subobjects or factor objects. Let $H: \mathcal{G} \rightleftarrows \mathcal{H} :T$ be a pair of functors with H a right adjoint of T, with unit $\sigma: 1 \to HT$ and counit $\rho: TH \to 1$. Then*:

(1) *if ρ_M is an isomorphism for every $M \in \mathcal{G}_0$, then T preserves exactness of all short exact sequences with objects in $H(\mathcal{G}_0)$*;

(2) *if σ_N is an isomorphism for every $N \in \mathcal{H}_0$, then H preserves exactness of all short exact sequences with objects in $T(\mathcal{H}_0)$.*

This lemma applies, in particular, when we are given a $\mathcal{C}_1$-$\mathcal{C}_2$-pretilting equivalence $F: \mathcal{T}_1 \rightleftarrows \mathcal{F}_2 :G$.

Since F is left exact and $\mathcal{C}_1$ has enough injective objects, we can define the right derived functors $F^{(i)}$, $i \geq 0$; we denote $F^{(1)}$ simply by F'.

1.5. PROPOSITION. *Let $F: \mathcal{T}_1 \rightleftarrows \mathcal{F}_2 :G$ be a $\mathcal{C}_1$-$\mathcal{C}_2$-tilting equivalence. Then*:

(1) $\mathcal{T}_1 = \ker F'$;

(2) $F^{(i)} = 0$, *for* $i \geq 2$.

(3) *for every* $M \in \mathcal{C}_1$, $M \in \mathcal{F}_1$ *if and only if* $F(M) = 0$;

(4) *for every* $M \in \mathcal{C}_1$, $F'(M) \in \mathcal{T}_2$.

Proof. Let $M \in \mathcal{C}_1$ and consider an exact sequence $0 \to M \to M_1 \to M_2 \to 0$. We can write the long exact sequence

$$0 \to F(M) \to F(M_1) \to F(M_2) \to F'(M) \to F'(M_1) \to F'(M_2)$$
$$\to F^{(2)}(M) \to \cdots \to F^{(n-1)}(M_2) \to F^{(n)}(M) \to \cdots$$

of the derived functors. Assume now that M_1 is injective. Then $M_1, M_2 \in \mathscr{T}_1$ and $F^{(i)}(M_1) = 0$, for all $i \geq 1$.

If $M \in \mathscr{T}_1$, then, by Lemma 1.4, the morphism $F(M_1) \to F(M_2)$ is epic, so that $F'(M) = 0$. Hence $\mathscr{T}_1 \subseteq \ker F'$ and $F^{(i)} = 0$, for $i \geq 2$, by induction.

Conversely, assume that $M \in \ker F'$. Then we can apply the functor G (which is right exact) to the exact sequence $0 \to F(M) \to F(M_1) \to F(M_2) \to 0$, getting the commutative diagram

$$\begin{tikzcd} & GF(M) \arrow[r] \arrow[d, "\rho_M"] & GF(M_1) \arrow[r] \arrow[d, "\rho_{M_1}"] & GF(M_2) \arrow[r] \arrow[d, "\rho_{M_2}"] & 0 \\ 0 \arrow[r] & M \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Since ρ_{M_1} and ρ_{M_2} are isomorphism, the "five lemma" shows that ρ_M is epic. Hence $M \in \mathscr{T}_1$, since $GF(M) \in \mathscr{T}_1$.

The definition of F tells us that $F(M) \cong F(t_1(M))$; hence $F(M) = 0$ if and only if $t_1(M) = 0$; i.e., $M \in \mathscr{F}_1$.

Consider now $M \in \mathscr{C}_1$: we have to show that $F'(M) \in \mathscr{T}_2$. Take the exact sequence $0 \to t_1 M \to M \to M/t_1 M \to 0$; applying it to the functor F, we get, by the above results, that $F'(M) \cong F'(M/t_1 M)$. We can thus assume $M \in \mathscr{F}_1$, so $F(M) = 0$. Thus, as above, we get an exact sequence $0 \to F(M_1) \to F(M_2) \to F'(M) \to 0$; applying G, we get the diagram

$$\begin{tikzcd} GF(M_1) \arrow[r] \arrow[d, "\rho_{M_1}"] & GF(M_2) \arrow[r] \arrow[d, "\rho_{M_2}"] & GF'(M) \arrow[r] & 0 \\ M_1 \arrow[r] & M_2 \arrow[r] & 0 & \end{tikzcd}$$

with exact rows. By the "five lemma," $GF'(M) = 0$ and so $F'(M) \in \mathscr{T}_2$. ∎

We now give a characterization of the objects in $\mathscr{F}_2$. Note that, if we could write the derived functors of G (i.e., if we had enough projective objects in $\mathscr{C}_2$), the condition would read: *$N \in \mathscr{F}_2$ if and only if $N \in \ker G'$.*

1.6. LEMMA. *Let $F: \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 : G$ be a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence and let $N \in \mathscr{C}_2$. The following conditions are equivalent*:

(a) $N \in \mathscr{F}_2$;

(b) *there exists an exact sequence* $0 \to N_2 \to N_1 \to N \to 0$, *with* $N_1, N_2 \in \mathscr{F}_2$, *such that the sequence* $0 \to GN_2 \to GN_1 \to GN \to 0$ *is exact*;

(c) *for any exact sequence* $0 \to N_2 \to N_1 \to N \to 0$, *with* $N_1, N_2 \in \mathscr{F}_2$, *the sequence* $0 \to GN_2 \to GN_1 \to GN \to 0$ *is exact.*

Proof. (a) ⇒ (c) Let $N \in \mathscr{F}_2$ and let

$$0 \to N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N \to 0$$

be an exact sequence with $N_1, N_2 \in \mathscr{F}_2$. The conclusion follows from Lemma 1.4.

(c) ⇒ (b) is obvious, since at least one such sequence exists.

(b) ⇒ (a) Applying F to the sequence $0 \to GN_2 \to GN_1 \to GN \to 0$ gives an exact sequence, since $GN_2 \in \mathscr{T}_1 = \ker F'$. Hence the canonical morphism $\sigma_N : N \to FGN$ is an isomorphism and $N \in \mathscr{F}_2 = \operatorname{im} F$. ∎

2. $*$-PAIRS OF FUNCTORS

Another approach to tilting equivalences is possible; the subject of this section stems from Menini and Orsatti [18] and Colpi [7]. I am grateful to Claudia Menini who allowed me to use some material developed by her.

For all this section we will assume the following situation: $\mathscr{C}_1$ and $\mathscr{C}_2$ are Grothendieck categories and

$$F : \mathscr{C}_1 \rightleftarrows \mathscr{C}_2 : G$$

is a pair of adjoint functors, with F a right adjoint of G. We denote as before by ρ and σ the counit and the unit of the adjunction

$$\rho_M : GFM \to M, \qquad \sigma_N : N \to FGN$$

for $M \in \mathscr{C}_1$ and $N \in \mathscr{C}_2$. Let $\mathscr{D}_1$ be the full subcategory of $\mathscr{C}_1$ consisting of those objects $M \in \mathscr{C}_1$ such that ρ_M is epic; analogously, let $\mathscr{D}_2$ be the full subcategory of $\mathscr{C}_2$ consisting of those objects $N \in \mathscr{C}_2$ such that σ_N is monic.

2.1. PROPOSITION. *The class $\mathscr{D}_1$ is closed under quotient objects and coproducts; the class $\mathscr{D}_2$ is closed under subjects and products. Moreover* $\operatorname{im} F \subseteq \mathscr{D}_2$ *and* $\operatorname{im} G \subseteq \mathscr{D}_1$.

Proof. If $f : M_1 \to M_2$ is epic, then $\rho_{M_2} \circ GFf = f \circ \rho_{M_1}$, so that ρ_{M_2} is epic if ρ_{M_1} is.

If M_λ is a family in $\mathscr{C}_1$, then it is easy to verify that the canonical morphism $\coprod GFM_\lambda \to GF(\coprod M_\lambda)$ composed with the counit is the coproduct of the counits: the diagram

$$\begin{tikzcd} \coprod GFM_\lambda \arrow[r] \arrow[dr] & GF(\coprod M_\lambda) \arrow[d] \\ & \coprod M_\lambda \end{tikzcd}$$

is commutative.

Let $N \in \mathscr{C}_2$; then $\rho_{GN} \circ G(\sigma_N)$ is the identity, so ρ_{GN} is a splitting epic. The other assertions are proved in a similar way. Note that σ_{FM} is a splitting monic, for all $M \in \mathscr{C}_2$. ▮

2.2. LEMMA. *Let $M \in \mathscr{C}_1$ and $N \in \mathscr{C}_2$; consider the canonical factorizations*

$$\begin{tikzcd} GFM \arrow[rr, "\rho_M"] \arrow[dr, "\bar{\rho}_M"'] & & M \\ & \operatorname{im}\rho_M \arrow[ur, "i_M"'] & \end{tikzcd} \qquad \begin{tikzcd} N \arrow[rr, "\sigma_N"] \arrow[dr, "\bar{\sigma}_N"'] & & FGN \\ & \operatorname{im}\sigma_N \arrow[ur, "j_N"'] & \end{tikzcd}$$

Then $\operatorname{im}\rho_M \in \mathscr{D}_1$, $\operatorname{im}\sigma_N \in \mathscr{D}_2$, *and Fi_M and Gj_N are isomorphisms.*

Proof. Since $\bar{\rho}_M$ is epic and $GFM \in \operatorname{im} G$, $\operatorname{im}\rho_M \in \mathscr{D}_1$, by the preceding proposition. Moreover F is left exact, so that Fi_M is monic. Moreover $F\rho_M \circ \sigma_{FM}$ is the identity, so $F\rho_M$ is epic; hence also Fi_M must be epic. In a similar way we conclude for the other parts. ▮

2.3. DEFINITION. The pair of functors (F, G) is called a $*$*-pair* if it induces an equivalence between $\mathscr{D}_1$ and $\mathscr{D}_2$. It is called a *strong* $*$*-pair* if, moreover, every object of $\mathscr{C}_1$ is a subobject of an object in $\mathscr{D}_1$ and every object of $\mathscr{C}_2$ is a quotient of an object in $\mathscr{D}_2$.

The name $*$-pair comes from the notion of a $*$-module. A module P_R is called a $*$*-module* if the functors $H = \operatorname{Hom}_R(P, \cdot)$ and $T = \cdot \otimes_S P$ induce an equivalence between $\operatorname{Gen}(P_R)$ (the class of P-generated R-modules) and $\operatorname{Cogen}(P_S^*)$ (the class of P^*-cogenerated S-modules), where $S = \operatorname{End}(P_R)$, W_R is an injective cogenerator of Mod-R, and $P^* = H(W)$. In this case $\mathscr{C}_1 =$ Mod-R, $\mathscr{C}_2 =$ Mod-S, $\mathscr{D}_1 = \operatorname{Gen}(P_R)$, and $\mathscr{D}_2 = \operatorname{Cogen}(P_S^*)$.

As is clear from the definition, a $*$-pair induces a "maximal equivalence," in the sense that $\mathscr{D}_1$ and $\mathscr{D}_2$ are the largest subcategories where F and G can induce an equivalence. We want to show that such maximal equivalences are just tilting equivalences, when we restrict the domains of the functors (see Theorem 2.8).

2.4. THEOREM. *The following statements are equivalent*:

(a) (F, G) *is a* $*$-*pair*;

(b) ρ_M *is monic for all* $M \in \mathscr{D}_1$ *and* σ_N *is epic for all* $N \in \mathscr{D}_2$;

(c) ρ_M *is monic for all* $M \in \mathscr{C}_1$ *and* σ_N *is epic for all* $N \in \mathscr{C}_2$.

Proof. We need only to prove (b) $\Rightarrow$ (c).

Let $M \in \mathscr{C}_2$ and set $M' = \operatorname{im} \rho_M$; we know from Lemma 2.2 that Fi_M is an isomorphism. As $i_M \circ \rho_{M'} = \rho_M \circ GFi_M$ is monic, then ρ_M is monic. In a similar way we conclude for the other parts. ∎

2.5. *Remark.* Denote by $\overline{\mathscr{D}}_1$ the full subcategory of $\mathscr{C}_1$ consisting of the subobjects of objects in $\mathscr{D}_1$; analogously, denote by $\overline{\mathscr{D}}_2$ the full subcategory of $\mathscr{C}_2$ consisting of the quotients of objects in $\mathscr{D}_2$. Then $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$ are Grothendieck categories and we can regard (F, G) as a pair of adjoint functors between $\overline{\mathscr{D}}_1$ and $\overline{\mathscr{D}}_2$.

2.6. PROPOSITION. *Let* (F, G) *be a* $*$-*pair and let the sequence* $0 \to M' \to M \to M'' \to 0$ *be exact, where* $M', M'' \in \mathscr{D}_1$ *and* $M \in \overline{\mathscr{D}}_1$. *Then* $M \in \mathscr{D}_1$.

Proof. Consider a monic $M \to M_1$, where $M_1 \in \mathscr{D}_1$; then form the push-out diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M_1 & \longrightarrow & M''' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_2 & = & M_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Apply the functor F and recall Lemma 1.4: the sequences $0 \to F(M'') \to F(M''') \to F(M_2) \to 0$ and $0 \to F(M') \to F(M_1) \to F(M''') \to 0$ are exact. Thus, by diagram chasing, the sequence $0 \to F(M) \to F(M_1) \to F(M_2) \to 0$ is exact and with objects in $\mathscr{D}_2$. Applying the functor G gives again an exact sequence and, by the "five lemma," ρ_M is an isomorphism. ∎

In the same way (but using pull-backs), we can prove the following result.

2.7. PROPOSITION. *Let* (F, G) *be a* $*$*-pair and let the sequence* $0 \to N' \to N \to N'' \to 0$ *be exact, where* $N', N'' \in \mathscr{D}_2$ *and* $N \in \overline{\mathscr{D}}_2$. *Then* $N \in \mathscr{D}_2$. ∎

The theory developed in this section allows us to apply the results in Section 1.

2.8. THEOREM. *Let* $F: \mathscr{C}_1 \rightleftarrows \mathscr{C}_2 : G$ *be a* $*$*-pair. Then this pair of functors induces a* $\overline{\mathscr{D}}_1$*-*$\overline{\mathscr{D}}_2$*-tilting equivalence*

$$F: \mathscr{D}_1 \rightleftarrows \mathscr{D}_2 : G.$$

Proof. By Propositions 2.6 and 2.7, the class $\mathscr{D}_1$ is a torsion class in $\overline{\mathscr{D}}_1$ and the class $\mathscr{D}_2$ is a torsion-free class in $\overline{\mathscr{D}}_2$. The other properties of a tilting equivalence follow by definition. ∎

This shows that the notions of a strong $*$-pair and of tilting equivalence are exactly the same.

3. BUILDING DERIVED FUNCTORS

Throughout this section we will fix a $\mathscr{C}_1$-$\mathscr{C}_1$-tilting equivalence $F: \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 : G$.

Our purpose will be to build the first left derived functor of $G: \mathscr{C}_2 \to \mathscr{C}_1$. The problem, of course, is that the category $\mathscr{C}_2$ need not have projective objects. However, for every $N \in \mathscr{C}_2$, there exists an $\mathscr{F}_2$*-presentation*

$$\mathscr{P}_N: \qquad 0 \to N_2 \xrightarrow{f_2} N_1 \xrightarrow{f_1} N \to 0,$$

where $N_1, N_2 \in \mathscr{F}_2 = \operatorname{im} F$. We recall also that a module N is in $\mathscr{F}_2$ if and only if applying G to any $\mathscr{F}_2$-presentation of N gives an exact sequence (see 1.6).

The construction of the left derived functor will follow this pattern: (1) given an $\mathscr{F}_2$-presentation of N, apply the functor G and take the kernel of Gf_2; (2) prove that (the domain of) this kernel is independent of the presentation, so that we will get a well defined action on objects; (3) define an action on morphisms, in order to get a functor.

To make notations shorter we will use symbols like before to denote $\mathscr{F}_2$-presentations (see, e.g., the following definition).

3.1. DEFINITION. Let $\mathscr{P}_N$ and $\mathscr{P}'_N$ be $\mathscr{F}_2$-presentations of N; we say that $\mathscr{P}_N$ is *over* $\mathscr{P}'_N$ if there exists a commutative diagram with exact rows

$$\begin{tikzcd}
\mathscr{P}_N\colon & 0 \arrow[r] & N_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & N_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & N \arrow[r] \arrow[d, equal] & 0 \\
\mathscr{P}'_N\colon & 0 \arrow[r] & N'_2 \arrow[r, "f'_2"] & N'_1 \arrow[r, "f'_1"] & N \arrow[r] & 0
\end{tikzcd}$$

where α_1 (hence also α_2) is epic.

3.2. PROPOSITION. *Let $\mathscr{P}'_N$ and $\mathscr{P}''_N$ be $\mathscr{F}_2$-presentations of N; then there exists an $\mathscr{F}_2$-presentation of $\mathscr{P}_N$ of N which is over $\mathscr{P}'_N$ and over $\mathscr{P}''_N$.*

Proof. Take for N_1 the pull-back of N'_1 and N''_1 over N; then complete with the kernel. ∎

Let $\mathscr{P}$ be an $\mathscr{F}_2$-presentation of N; then we set $\mathscr{K}(\mathscr{P}) = \ker Gf_2$. If $\mathscr{P}$ is over $\mathscr{P}'$, then we get a unique morphism $\tilde{\alpha}$ making commutative the following diagram (with exact rows):

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K}(\mathscr{P}_N) \arrow[r] \arrow[d, "\tilde{\alpha}"] & GN_2 \arrow[r, "Gf_2"] \arrow[d, "G\alpha_2"] & GN_1 \arrow[r, "Gf_1"] \arrow[d, "G\alpha_1"] & GN \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{K}(\mathscr{P}'_N) \arrow[r] & GN'_2 \arrow[r, "Gf'_2"] & GN'_1 \arrow[r, "Gf'_1"] & GN \arrow[r] & 0
\end{tikzcd}$$

3.3. LEMMA. *The morphism $\tilde{\alpha}$ of the preceding diagram is an isomorphism.*

Proof. We can form the following commutative diagram, with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N_2 \arrow[r, equal] \arrow[d, "k"'] & N_2 \arrow[d, "f_2"] & \\
0 \arrow[r] & N'_2 \arrow[r, "h"] \arrow[d, equal] & X \arrow[r, "q"] \arrow[d, "p"'] & N_1 \arrow[r] \arrow[d, "f_1"] & 0 \\
0 \arrow[r] & N'_2 \arrow[r, "f'_2"'] & N'_1 \arrow[r, "f'_1"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where X is the pull-back of f_1 and f'_1. Now the properties of the pull-back guarantee that there exists a unique morphism $q'\colon N_1 \to X$ such that qq' is the identity on N_1 and $pq' = \alpha_1$. Hence the middle row splits and there exists a unique morphism $h'\colon X \to N'_2$ such that $h'h$ and $hh' + q'q$ are the identities on N'_2 and X, respectively.

We want to show that $h'k = -\alpha_2$. Using MacLane's trick of *members* (see [16]), let $x \in N_2$; then $k(x) = q'qk(x) + hh'k(x) = q'f_2(x) + hh'k(x)$; moreover

$$0 = pk(x) = pq'f_2(x) + phh'k(x) = \alpha_1 f_2(x) + f'_2 h'k(x).$$

Thus $f'_2 h'k(x) = -\alpha_1 f_2(x) = -f'_2 \alpha_2(x)$, so that $h'k(x) = -\alpha_2(x)$.

Apply now G to the preceding diagram; taking kernels, we get the commutative diagrams with exact rows and columns:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & 0 \arrow[d] & \ker Gf_2 \arrow[d] & \\
 & & & GN_2 \arrow[r, equal] \arrow[d, "Gk"'] & GN_2 \arrow[d, "Gf_2"] & \\
 & 0 \arrow[r] & GN'_2 \arrow[r, "Gh"] \arrow[d] & GX \arrow[r, "Gq"] \arrow[d, "Gp"'] & GN_1 \arrow[r] \arrow[d, "Gf_1"] & 0 \\
0 \arrow[r] & \ker Gf'_2 \arrow[r] & GN'_2 \arrow[r, "Gf'_2"'] & GN'_1 \arrow[r, "Gf'_1"'] \arrow[d] & GN \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 &
\end{tikzcd}$$

Proceeding as in the "snake lemma," we can define a morphism $d\colon \mathscr{K}(\mathscr{P}) = \ker Gf_2 \to \mathscr{K}(\mathscr{P}') = \ker Gf'_2$ by setting $d(t) = t'$ if and only if $Gh(t') = Gk(t)$. By symmetry, it is clear that d is an isomorphism. But now, for $t \in \ker Gf_2$, we have

$$d(t) = Gh'Gh(d(t)) = Gh'Gk(t) = -G\alpha_2(t);$$

hence $-d = \tilde{\alpha}$, by uniqueness of $\tilde{\alpha}$, and so $\tilde{\alpha}$ is an isomorphism. ∎

In this way we have covered steps (1) and (2). It remains to define the action on morphisms. The above considerations allow us to choose whatever $\mathscr{F}_2$-presentation of a module we need. We can thus define, for $N \in \mathscr{C}_2$,

$$G'N = \mathscr{K}(\mathscr{P}_N),$$

where $\mathscr{P}_N$ is a fixed $\mathscr{F}_2$-presentation of N. It is also clear that $G'N = 0$ if and only if $N \in \mathscr{F}_2$.

Let $f\colon M \to N$ be a morphism in $\mathscr{C}_2$. Then we can find $\mathscr{F}_2$-presentations $\mathscr{P}_M$ and $\mathscr{P}_N$ of M and N, respectively, and a commutative diagram

$$\begin{array}{ccccccccc} \mathscr{P}_M\colon & 0 & \longrightarrow & M_2 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow 0 \\ & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & \\ \mathscr{P}_N\colon & 0 & \longrightarrow & N_2 & \longrightarrow & N_1 & \longrightarrow & N & \longrightarrow 0 \end{array}$$

to which we can apply G, take kernels, and get a morphism $G'f\colon G'M \to G'N$. The existence of such a diagram can be stated in the following way: take an arbitrary $\mathscr{F}_2$-presentation for M,

$$0 \to M_2' \overset{g_2'}{\to} M_1' \overset{g_1'}{\to} M \to 0,$$

and form the pull-back

$$\begin{array}{ccc} M_1 & \longrightarrow & M_1' \\ \downarrow & & \downarrow{\scriptstyle fg_1'} \\ N_1 & \overset{f_1}{\longrightarrow} & N, \end{array}$$

where $M_1 \in \mathscr{F}_2$; then take M_2 as the kernel of $M_1 \to M$. It is easy to verify that this defines a functor $G'\colon \mathscr{C}_2 \to \mathscr{C}_1$.

3.4. THEOREM. *Let $F\colon \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 \colon G$ be a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence. Then there exists a functor $G'\colon \mathscr{C}_2 \to \mathscr{C}_1$ which is a right adjoint to F'. Moreover*:

(1) *the image of G' is $\mathscr{F}_1$*;

(2) *G' commutes with coproducts*;

(3) *the restrictions of the functors F' and G' define an equivalence between $\mathscr{F}_1$ and $\mathscr{T}_2$*;

(4) *for every exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}_2$, there exists a morphism $\partial\colon G'(N) \to G(L)$ such that the sequence*

$$0 \to G'(L) \to G'(M) \to G'(N) \overset{\partial}{\to} G(L) \to G(M) \to G(N) \to 0$$

is exact;

(5) $\mathscr{F}_2 = \ker G'$.

Proof. Define G' as above; take $N \in \mathscr{T}_2$, and fix an $\mathscr{F}_2$-presentation $\mathscr{P}_N$; since $GN = 0$, we get the exact sequence $0 \to G'N \to GN_2 \to GN_1 \to$

0 and so the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_2 & \longrightarrow & N_1 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow FG'N & \longrightarrow & FGN_2 & \longrightarrow & FGN_1 & \longrightarrow & F'G'N & \longrightarrow & 0
\end{array}$$

where the vertical arrows are isomorphisms. Hence $FG'N = 0$ and $G'N \in \mathscr{F}_1$; moreover this defines a unique isomorphism $\alpha_N : N \to F'G'N$.

Take $M \in \mathscr{F}_1$ and an exact sequence $0 \to M \to M_1 \to M_2 \to 0$, where $M_1 \in \mathscr{T}_1$. Then we get an exact sequence

$$0 \to FM_1 \to FM_2 \to F'M \to 0$$

which is an $\mathscr{F}_2$-presentation of $F'M \in \mathscr{T}_2$; thus we can write a diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G'F'M & \longrightarrow & GFM_1 & \longrightarrow & GFM_2 & \longrightarrow & 0
\end{array}$$

and we get a unique isomorphism $\beta_M : M \to G'F'M$. The uniqueness of these isomorphisms shows that they are natural, so we have an equivalence $\mathscr{F}_1 \rightleftarrows \mathscr{T}_2$ induced by F' and G'.

The fact that G' commutes with coproducts follows easily, since a coproduct of $\mathscr{F}_2$-presentations is an $\mathscr{F}_2$-presentation of the coproduct and G commutes with coproducts, being a left adjoint.

We now consider the torsion functors t_1 and t_2; it is clear that, for $M \in \mathscr{C}_1$ and $N \in \mathscr{C}_2$,

$$F'M \cong F'\big(M/t_1(M)\big) \qquad \text{and} \qquad G'N \cong G'\big(t_2(N)\big)$$

canonically; in particular $\mathscr{F}_2 = \ker G'$. We have thus showed that $G' : \mathscr{T}_2 \rightleftarrows \mathscr{F}_1 : F'$ is a pretilting equivalence, so we can apply Proposition 1.3, where G' replaces F and F' replaces G. Moreover G' and F' are just the extensions mentioned in the proof of that proposition, so that G' is a right adjoint to F'.

It remains to prove the existence of a connecting morphism; assume we are given an exact sequence in $\mathscr{C}_2$

$$0 \to L \to M \to N \to 0;$$

We can embed it into a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_2 & \longrightarrow & M_2 & \longrightarrow & N_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are $\mathscr{F}_2$-presentations. We can apply G to this diagram, recalling that $G'L$ is the kernel of $GL_2 \to GL_1$, and the same for the others: a standard application of the "snake lemma" (e.g., [22, Corollary 4.11.9]) gives the connecting morphism we are looking for. ∎

The next proposition shows the behavior of G with respect to finitely generated objects. Recall that an object M in a Grothendieck category $\mathscr{C}$ is *small* if the functor $\operatorname{Hom}_{\mathscr{C}}(M, \cdot)\colon \mathscr{C} \to \mathbf{Ab}$ commutes with coproducts. Every finitely generated object is small, but the converse is, in general, false.

3.5. *Remark*. The functor F commutes with coproducts, since both the classes $\mathscr{T}_1$ and $\mathscr{F}_2$ are closed under coproducts.

3.6. PROPOSITION. *Let $F\colon\mathscr{T}_1 \rightleftarrows \mathscr{F}_2 \colon G$ be a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence. Then, for every small object $N \in \mathscr{C}_2$, GN is small.*

Proof. We can use the adjunction between F and G and the fact that N is small. Indeed, let (X_λ) be a family of objects in $\mathscr{C}_1$; then

$$\begin{aligned}
\operatorname{Hom}\Bigl(GN, \coprod_\lambda X_\lambda\Bigr) &\cong \operatorname{Hom}\Bigl(N, F\Bigl(\coprod_\lambda X_\lambda\Bigr)\Bigr) && \text{(adjunction)} \\
&\cong \operatorname{Hom}\Bigl(N, \coprod_\lambda FX_\lambda\Bigr) && \text{(Remark 3.5)} \\
&\cong \coprod_\lambda \operatorname{Hom}(N, FX_\lambda) && (N \text{ is small}) \\
&\cong \coprod_\lambda \operatorname{Hom}(GN, X_\lambda) && \text{(adjunction)}
\end{aligned}$$

and the thesis follows. ∎

We can collect everything we have done in a "tilting theorem" (notations are as at the beginning of this section).

3.7. TILTING THEOREM. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Grothendieck categories and let $F:\mathscr{T}_1 \rightleftarrows \mathscr{F}_2 :G$ be a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence. Then*:

(1) *there exists a left derived functor G' of G and G' is left exact*;

(2) *G' is a right adjoint to the first derived functor F' of F*;

(3) *the right derived functors $F^{(i)}$ of F are zero, for $i \geq 2$*;

(4) *the functors F' and G' induce an equivalence between $\mathscr{F}_1$ and $\mathscr{T}_2$*;

(5) *$F'G$ and $G'F$ are zero functors*;

(6) *for any object $M \in \mathscr{C}_1$, $t_1 M \cong GFM$ and $M/t_1 M \cong G'F'M$, so there exists an exact sequence*

$$0 \to GFM \to M \to G'F'M \to 0;$$

(7) *for any object $N \in \mathscr{C}_2$, $t_2 N \cong F'G'N$ and $N/t_2 N \cong FGN$, so there exists an exact sequence*

$$0 \to F'G'N \to N \to FGN \to 0.$$

(8) $\mathscr{T}_1 = \ker F'$, $\mathscr{F}_1 = \ker F$, $\mathscr{T}_2 = \ker G$, *and* $\mathscr{F}_2 = \ker G'$.

We add finally a characterization of the classes involved in the tilting theorem. If X is an object in a Grothendieck category, we denote by:

(a) $\operatorname{Gen}(X)$ the class of objects that are *generated* by X, i.e., all epimorphic images of coproducts of copies of X;

(b) $\operatorname{Pres}(X)$ the class of objects that are *presented* by X, i.e., all objects M admitting an exact sequence of the form

$$X^{(\alpha)} \to X^{(\beta)} \to M \to 0;$$

(c) $\operatorname{Cogen}(X)$ the class of objects that are *cogenerated* by X, i.e., all subobjects of products of copies of X;

(d) $\operatorname{Copres}(X)$ the class of objects that are *copresented* by X, i.e., all objects M admitting an exact sequence of the form

$$0 \to M \to X^{\alpha} \to X^{\beta}.$$

3.8. THEOREM. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Grothendieck categories and let $F:\mathscr{T}_1 \rightleftarrows \mathscr{F}_2 :G$ be a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence. Assume that X is a cogenerator of $\mathscr{C}_1$ and that Y is a generator of $\mathscr{C}_2$. Then*:

(1) $\mathscr{T}_1 = \operatorname{Gen}(G(Y)) = \operatorname{Pres}(G(Y))$;

(2) $\mathscr{F}_2 = \operatorname{Cogen}(F(X)) = \operatorname{Copres}(F(X))$.

Proof. Since $G(Y) \in \mathcal{T}_1$, by the closure properties of $\mathcal{T}_1$ it is sufficient to show that $\mathcal{T}_1 \subseteq \mathrm{Pres}(G(Y))$. Let $M \in \mathcal{T}_1$; then there exists an exact sequence $Y^{(\alpha)} \to Y^{(\beta)} \to FM \to 0$ in $\mathcal{C}_2$ and the conclusion follows, since G is right exact. The other statement can be proved similarly. ∎

4. RINGS WITH LOCAL UNITS

Rings with *local units* were introduced by Ánh and Márki [1] as those rings S not necessarily with unit in which, for any finite set $\{s_1, s_2, \dots, s_n\}$ there exists an idempotent $e \in S$ such that

$$es_i = s_i = s_i e \qquad (i = 1, 2, \dots, n).$$

Examples of rings with local units are rings *with enough idempotents*, defined by Fuller [11]. If S is a ring with local units (for short an *lu-ring*) and X is any set, then the ring of *finite matrices over* X, i.e., the ring of all X-indexed matrices in which all but a finite number of entries are zero, is again an lu-ring.

A right module N_S over an lu-ring S is called *unitary* if $NS = S$. In all this section we will only consider unitary modules over a fixed lu-ring S. The category Mod-S of right unitary modules over S is a Grothendieck category with enough projective objects. Indeed S_S is a generator and, for any idempotent $e \in S$, eS is a finitely generated projective module.

We want now to consider a $\mathcal{C}_1$-$\mathcal{C}_2$-tilting equivalence where $\mathcal{C}_2 = \text{Mod-}S$. To save on notations, we set $\mathcal{C}_1 = \mathcal{C}$, $\mathcal{T}_1 = \mathcal{T}$, $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{T}_2 = \mathcal{T}_S$, $\mathcal{F}_2 = \mathcal{F}_S$ and speak about a *$\mathcal{C}$-S-tilting equivalence* $F\colon \mathcal{T} \rightleftarrows :\mathcal{F}_S$. This setting will remain fixed up to Definition 4.5.

4.1. LEMMA. *The module S belongs to $\mathcal{F}_S$.*

Proof. The map

$$S \to \prod_{e = e^2 \in S} eS$$

is injective (note that the product is taken in the category Mod-S). ∎

If $M, M' \in \mathcal{C}$, then $\mathrm{Hom}(M, M')$ is the group of $\mathcal{C}$-morphisms from M to M' and $\mathrm{End}(M)$ denotes the endomorphism ring of M.

4.2. LEMMA. *Let $P = G(S_S)$; then $\mathcal{T} = \mathrm{Gen}(P) = \mathrm{Pres}(P)$ and S embeds canonically in* $\mathrm{End}(P)$.

Proof. The first statement follows from Theorem 3.8. The second one is an easy consequence of the equivalence and the facts that $S \in \mathcal{F}_S$ and S embeds in $\mathrm{End}(S_S)$. ∎

The preceding lemma allows us to define a functor $H = \mathrm{Hom}(P, \cdot)S: \mathscr{C} \to \text{Mod-}S$, in a similar way as in [1, p. 3]. Indeed, for $M \in \mathscr{C}$, the Abelian group $\mathrm{Hom}(P, M)$ carries a natural structure of a right S-module, possibly non-unitary. Then

$$H(M) = \mathrm{Hom}(P, M)S$$

is the largest unitary submodule of $\mathrm{Hom}(P, M)$. It is straightforward to extend this to a functor $\mathscr{C} \to \text{Mod-}S$.

4.3. PROPOSITION. *The functor F is naturally isomorphic to H.*

Proof. Let $M \in \mathscr{C}$; then $FM \cong \mathrm{Hom}_S(S, FM)S$ canonically by [1, Proposition 1.1] and so we have the chain of natural isomorphisms

$$FM \cong \mathrm{Hom}_S(S, FM)S \cong \mathrm{Hom}(GS, M)S = \mathrm{Hom}(P, M)S$$

and we are done. ▮

By Lemma 4.2, we can view each element of S as an endomorphism of P; if $s \in S$, we put $sP = \operatorname{im} s$ and $[s]P = \ker s$. When $e = e^2 \in S$, we have a direct decomposition $P = eP \oplus [e]P$. More generally, when $X \subseteq S$, we consider $XP = \Sigma\{\operatorname{im} s = sP : s \in X\}$.

4.4. PROPOSITION. *Let $e = e^2 \in S$; then eP is canonically isomorphic to $G(eS)$. Moreover $SP = P$.*

Proof. Let us prove that $eP \cong G(eS)$. Since $eP \in \mathrm{Gen}(P) = \mathscr{T}$, we can consider the canonical embedding $j_e: eP \to P$; then Hj_e is an injection $H(eP) \to H(P) = S$ and it is easy to see that $\operatorname{im} Hj_e \subseteq eS$. If $s = es \in eS$, then s, seen as an endomorphism of P, factors through eP. Hence $H(eP) = eS$.

Consider now the epimorphism

$$\bigoplus_{e = e^2 \in S} eS \to S \to 0.$$

Applying G, we get an epimorphism $\coprod_e G(eS) \to P$ and thus an epimorphism $\coprod_e eP \to P$. This implies that $P = SP$. ▮

The last thing we note is that eP is small, since eS is finitely generated, by Proposition 3.6. Putting together these facts, Theorem 3.7, and Theorem 3.8, we have the following proposition, in which we denote by H' the first derived functor of H.

4.5. PROPOSITION. *Let $\mathscr{C}$ be a Grothendieck category, let S be an lu-ring, and let $F:\mathscr{T} \rightleftarrows \mathscr{F}_S :G$ be a $\mathscr{C}$-S-tilting equivalence. Then there exists an object $P \in \mathscr{C}$ such that*:

(1) *S embeds in* $\operatorname{End}(P)$ *and $SP = P$*;

(2) *F is canonically isomorphic to $H = \operatorname{Hom}(P, \cdot\,)S$*;

(3) *$\mathscr{T} = \operatorname{Gen}(P) = \ker H'$ and eP is small, for any idempotent $e \in S$.*

We now want to show that the converse of this result holds, thus generalizing [8, Theorem 4.1].

4.6. DEFINITION. Let $\mathscr{C}$ be a Grothendieck category and let $P \in \mathscr{C}$; let S be a subring of $\operatorname{End}(P)$ (not necessarily with unit) so that $\operatorname{Hom}(P, \cdot\,)S$ is a functor $\mathscr{C} \to \text{Mod-}S$. We say that (P, S) is a *tilting pair* in $\mathscr{C}$ if

(a) $SP = P$ and S is an lu-ring;

(b) eP is small, for every idempotent $e \in S$;

(c) $\operatorname{Gen}(P) = \ker H'$, where H' is the first derived functor of $\operatorname{Hom}(P, \cdot\,)S$.

Note that, when $S = \operatorname{End}(P)$, this definition coincides with that of a tilting object in $\mathscr{C}$ [8, Definition 2.3].

4.7. LEMMA. *Let (P, S) be a tilting pair in $\mathscr{C}$. Then* $\operatorname{Gen}(P) = \operatorname{Pres}(P)$ *and $H = \operatorname{Hom}(P, \cdot\,)S$ commutes with coproducts.*

Proof. We can apply the technique used in [10, Lemma 1.2] to get the equality $\operatorname{Gen}(P) = \operatorname{Pres}(P)$. Let $M \in \operatorname{Gen}(P)$; then, setting $X = \operatorname{Hom}(P, M)S$ as a set, we have an exact sequence

$$0 \to K \to \coprod_{\alpha \in X} e_\alpha P \xrightarrow{\beta} M \to 0,$$

where, for $\alpha \in X$, e_α is an idempotent in S such that $\alpha e_\alpha = \alpha$; thus the restriction of α to $e_\alpha P$ has the same image as α. The morphism β is the codiagonal. If we apply H to this sequence, we get that $H(\beta)$ is surjective and so $H'(K) = 0$.

Finally, if $\alpha \in \operatorname{Hom}(P, M)S$, its image is the same as that of the restriction of α to $e_\alpha P$, which is small; therefore H commutes with coproducts. ∎

Note that P is also a subgenerator of $\mathscr{C}$ (i.e., every $M \in \mathscr{C}$ is a subobject of some $M' \in \operatorname{Gen}(P)$), since every injective object is in $\operatorname{Gen}(P)$. Moreover, for every object $M \in \operatorname{Gen}(P)$, there exists an epimorphism $\coprod_\lambda e_\lambda P \to M$, where (e_λ) is a family of idempotents in S.

4.8. THEOREM. *Let $\mathscr{C}$ be a Grothendieck category and let (P, S) be a tilting pair in $\mathscr{C}$. Denote by H the functor* $\operatorname{Hom}(P, \cdot)S: \mathscr{C} \to \text{Mod-}S$ *and let W be an injective cogenerator of $\mathscr{C}$. Then*:

(1) *H has a left adjoint $T: \text{Mod-}S \to \mathscr{C}$*;

(2) $(\operatorname{Gen}(P), \ker H)$ *is a torsion theory in $\mathscr{C}$ and every injective object in $\mathscr{C}$ belongs to* $\operatorname{Gen}(P)$;

(3) $(\ker T, \operatorname{Cogen}(HW))$ *is a torsion theory in* Mod-S *and every projective object in* Mod-S *belongs to* $\operatorname{Cogen}(HW)$;

(4) *$H: \operatorname{Gen}(P) \rightleftarrows \operatorname{Cogen}(HW): T$ is a $\mathscr{C}$-S-tilting equivalence.*

Proof. (2) [See 8, Sect. 2]. We denote by t_P the torsion radical associated to this torsion theory: for all $M \in \mathscr{C}$,

$$t_P(M) = \sum \{\operatorname{im} \alpha \mid \alpha \in \operatorname{Hom}(eP, M),\, e = e^2 \in S\},$$

since $SP = P$.

Let us prove (1). For every $e = e^2 \in R$, e induces a direct decomposition of M: $P = eP \oplus [e]P$. Let us call $\mathscr{P}$ the subcategory consisting of the S-modules eS, for $e = e^2 \in S$. It is clear that $\mathscr{P}$ is a family of projective generators of Mod-S.

We want to define a functor $T_0: \mathscr{P} \to \mathscr{C}$; the action on objects is obvious: $T_0(eS) = eP$, which is well defined, as $eS = fS$ implies $eP = fP$. Next, a morphism $\alpha: eS \to fS$ is the left multiplication by an element of fS. Thus it induces a morphism $P \to fP$, whose restriction to eP we denote by $T_0(\alpha)$, and this is a good definition of a functor.

By [2, Theorem 3.6.2], T_0 extends uniquely to a functor $T: \text{Mod-}S \to \mathscr{C}$, which is right exact and commutes with filtered colimits, thus also with coproducts, and is right exact [22, Proposition 3.2.10]. We have to prove that T is a left adjoint of H.

The unit, i.e., a functorial morphism from the identity to HT, can be built starting, for $N \in \text{Mod-}S$, from an exact sequence

$$S'' = \bigoplus_{\mu} f_\mu S \xrightarrow{\beta} S' = \bigoplus_{\lambda} e_\lambda S \xrightarrow{\alpha} N \to 0.$$

We apply T, obtaining the exact sequence

$$T(S'') \xrightarrow{T\beta} T(S') \xrightarrow{T\alpha} T(N) \to 0;$$

set $K = \operatorname{im} T\beta \in \operatorname{Gen}(P)$ and let $\gamma: T(S'') \to K$ be the induced epimorphism. By the fact that $T(eS) = eP$ and that eP is small, there exist isomorphisms $\sigma_{S'}: S' \to HT(S')$ and $\sigma_{S''}: S'' \to HT(S'')$. Applying H to the exact sequence

$$0 \to K \to T(S') \to T(N) \to 0$$

yields an exact sequence, since $K \in \operatorname{Gen}(P)$. So we obtain the diagram with exact rows

$$\begin{tikzcd} S'' \arrow[r, "\beta"] \arrow[d, "H\gamma \circ \sigma_{S''}"'] & S' \arrow[r, "\alpha"] \arrow[d, "\sigma_{S'}"', "\cong"] & N \arrow[r] \arrow[d, dashed, "\sigma_N"] & 0 \\ H(K) \arrow[r] & HT(S') \arrow[r] & HT(N) \arrow[r] & 0 \end{tikzcd}$$

which can be completed in a unique way with a morphism $\sigma_N \colon N \to HT(N)$. This is easily seen to give the required unit; moreover, by the "five lemma," σ_N is epic, for every $N \in \text{Mod-}S$.

If $\mathscr{B}$ is a class of objects in a Grothendieck category $\mathscr{A}$, we denote by $\mathscr{B}_\oplus$ the full subcategory of $\mathscr{A}$ whose objects are coproducts of objects in $\mathscr{B}$. In particular, if we consider $\mathscr{P}$ as before and $\mathscr{Q} = \{eP : e = e^2 \in S\}$, then the functors T and H induce functors $\mathscr{P}_\oplus \to \mathscr{Q}_\oplus$ and $\mathscr{Q}_\oplus \to \mathscr{P}_\oplus$, respectively. By the fact that both functors commute with coproducts, it is clear that we get an adjunction, where the unit is the same as before; we denote by ρ the counit.

We want to build a counit ρ from TH to the identity. Let $M \in \mathscr{C}$; then $H(M) = H(t_P(M))$, so it is sufficient to define ρ_M for $M \in \operatorname{Gen}(P)$. Since $\operatorname{Gen}(P) = \operatorname{Pres}(P)$, there exists an exact sequence

$$0 \to K \xrightarrow{\beta} P' \xrightarrow{\alpha} M \to 0,$$

where $K \in \operatorname{Gen}(P)$ and $P' \in \mathscr{Q}_\oplus$. By the same fact there exists another exact sequence

$$0 \to K' \xrightarrow{\delta} P'' \xrightarrow{\gamma} K \to 0,$$

where $K' \in \operatorname{Gen}(P)$ and $P'' \in \mathscr{Q}_\oplus$. If we apply H to both sequences, we get again exact sequences and so we can apply T, which is right exact. This leaves us with the diagrams with exact rows

$$\begin{tikzcd} TH(K) \arrow[r, "TH\beta"] & TH(P') \arrow[r, "TH\alpha"] \arrow[d, "\rho_{P'}"', "\cong"] & TH(M) \arrow[r] & 0 \\ K \arrow[r, "\beta"] & P' \arrow[r, "\alpha"] & M \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} TH(K') \arrow[r, "TH\delta"] & TH(P'') \arrow[r, "TH\gamma"] \arrow[d, "\rho_{P''}"', "\cong"] & TH(K) \arrow[r] & 0 \\ K' \arrow[r, "\delta"] & P'' \arrow[r, "\gamma"] & K \arrow[r] & 0 \end{tikzcd}$$

which we can combine into one:

$$
\begin{tikzcd}
TH(P'') \arrow[r, "TH(\beta\gamma)"] \arrow[d, "\rho_{P''}"', "\cong"] & TH(P') \arrow[r, "TH\alpha"] \arrow[d, "\rho_{P'}"', "\cong"] & TH(M) \arrow[r] \arrow[d, dashed, "\rho_M"'] & 0 \\
P'' \arrow[r, "\beta\gamma"] & P' \arrow[r, "\alpha"] & M \arrow[r] & 0
\end{tikzcd}
$$

The leftmost rectangle commutes because of the adjunction $\mathscr{Q} \rightleftarrows \mathscr{P}$. Thus there exists a unique ρ_M making the diagram commute. Standard calculations show that this is the counit we were looking for. Moreover, the "five lemma" shows that ρ_M is an isomorphism.

The remaining parts are proved by noting that (H, T) is a strong $*$-pair and by observing that there is no problem in producing from the beginning the ith left derived functor of T, since the category Mod-S has enough projective objects. We have a formula like the one in [8, Lemma 1.4], which is derived from a more general formula in [23, Theorem 11.40],

$$\operatorname{Ext}_S^i(\cdot, HW) \cong \operatorname{Hom}\big(T_{(i)}(\cdot), W\big)$$

(this is why we take W injective). This will easily give the fact that $\operatorname{Cogen}(HW) = \ker T'$. The facts that we need (see Theorem 2.4) have already been proved, namely that ρ_M is monic for every $M \in \mathscr{C}$ and that σ_N is epic, for every $N \in$ Mod-S. ∎

5. MISCELLANEOUS ASPECTS

Tilting modules are very important in the representation theory of finite dimensional algebras. Therefore we want to show some properties of general tilting equivalences with respect to finitely generated objects. We will limit the discussion to the case when the categories are locally noetherian, reserving the more general context to the statement of an open problem.

A Grothendieck category $\mathscr{C}$ is called *locally noetherian* if it has a set of generators consisting of noetherian objects.

The following proposition is a generalization of [8, Lemma 6.1].

5.1. THEOREM. *Suppose we are given a $\mathscr{C}_1$-$\mathscr{C}_2$-tilting equivalence $F: \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 : G$, where $\mathscr{C}_1$ and $\mathscr{C}_2$ are locally noetherian. Then, for every noetherian object $M \in \mathscr{C}_1$ and $N \in \mathscr{C}_2$, FM, $F'M$, GN, and $G'N$ are noetherian.*

Proof. It follows from the assumptions on $\mathscr{F}_2$ that there exists a set of noetherian generators for $\mathscr{C}_2$ consisting of objects in $\mathscr{F}_2$. Hence there exists an exact sequence $0 \to N_2 \to N_1 \to N \to 0$, where $N_1, N_2 \in \mathscr{F}_2$ are noetherian. Applying G we get the exact sequence

$$0 \to G'N \to GN_2 \to GN_1 \to GN \to 0,$$

since $G'N_1 = 0$. Now we know that GN_1 and GN_2 are small; we can use a similar technique as in [9, proof of Proposition 1.9] to get that they are finitely generated, hence noetherian. Indeed, the fact that is needed is that a coproduct of injective objects in $\mathscr{C}$ is still injective; this holds in any locally noetherian Grothendieck category.

Take now an epimorphism $f\colon \coprod_\lambda N_\lambda \to FM$, where every $N_\lambda \in \mathscr{F}_2$ is noetherian, and let X be its kernel. Then the sequence $0 \to GX \to G(\coprod_\lambda N_\lambda) \to M \to 0$ is exact and so Gf factors through a finite directs sum; hence we have a sequence $0 \to Y \to \coprod_{i=1}^n GN_{\lambda_i} \to GFM \to 0$, where the terms are noetherian. By applying F we get the exact sequence

$$0 \to FY \to \coprod_{i=1}^{n} N_{\lambda_i} \to FM \to F'Y \to 0.$$

Then we need only to prove that $F'M$ is noetherian, for every noetherian $M \in \mathscr{C}_1$. We need only to show that $F'M$ is small. Now, for any family of objects $X_\lambda \in \mathscr{C}_2$,

$$\begin{aligned}
\operatorname{Hom}\Big(F'M, \coprod_\lambda X_\lambda\Big) &\cong \operatorname{Hom}\Big(M, G'\Big(\coprod_\lambda X_\lambda\Big)\Big) && \text{(adjunction)}\\
&\cong \operatorname{Hom}\Big(M, \coprod_\lambda G'X_\lambda\Big) && \text{(Theorem 3.4(2))}\\
&\cong \coprod_\lambda \operatorname{Hom}(M, G'X_\lambda) && (M \text{ is noetherian})\\
&\cong \coprod_\lambda \operatorname{Hom}(F'M, X_\lambda) && \text{(adjunction)}
\end{aligned}$$

and the thesis follows. ∎

We end with an open problem on tilting equivalences. It is well known that any tilting module is finitely generated (beware: there are many definitions of tilting modules in the literature; we opt to consider only tilting modules giving tilting equivalences, i.e., the modules that Colpi and Trlifaj [10] call *classical tilting modules*).

The problem is the following: *Assume a* $\mathscr{C}_1$-$\mathscr{C}_2$-*tilting equivalence* $F\colon \mathscr{T}_1 \rightleftarrows \mathscr{F}_2 : G$ *is given*; *is it true that* G *takes finitely generated objects of* $\mathscr{F}_2$ *to finite generated objects*?

Two results could provide a positive answer: (1) G' *commutes with filtered colimits* or (2) F *commutes with filtered colimits of objects in* $\mathscr{T}_1$.

REFERENCES

1. P. N. Ánh and L. Márki, Morita equivalence for rings without identity, *Tsukuba J. Math.* **11** (1987), 1–16.
2. I. Assem, "Tilting theory—An introduction," Topics in algebra, Part 1, Banach Center Publ. No. 26, Part 1, PWN, Warsaw, 1990.
3. K. Bongartz, Tilted algebras, *in* "Representations of Algebras (Puebla, 1980)," pp. 26–38, Springer-Verlag, Berlin, 1981.
4. S. Brenner and M. C. R. Butler, Generalizations of the Bernstein–Gel'fand–Ponomarev reflection functors, *in* "Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)," pp. 103–169, Springer-Verlag, Berlin, 1980.
5. R. R. Colby and K. R. Fuller, Tilting, cotilting, and serially tilted rings, *Comm. Algebra* **18** (1990), 1585–1615.
6. R. R. Colby and K. R. Fuller, Tilting and torsion theory counter equivalences, *Comm. Algebra* **23** (1995), 4833–4949.
7. R. Colpi, Tilting modules and $*$-modules, *Comm. Algebra* **21** (1993), 1095–1102.
8. R. Colpi, Tilting in Grothendieck categories, *Forum Math.* **11** (1999), 735–759.
9. R. Colpi and C. Menini, On the structure of $*$-modules, *J. Algebra* **158** (1993), 400–419.
10. R. Colpi and J. Trlifaj, Tilting modules and tilting torsion theories, *J. Algebra* **178** (1995), 614–634.
11. K. R. Fuller, On rings whose left modules are direct sums of finitely generated modules, *Proc. Amer. Math. Soc.* **54** (1976), 39–44.
12. E. Gregorio, Generalized Morita equivalence for linearly topologized rings, *Rend. Sem. Mat. Univ. Padova* **79** (1988), 221–246.
13. E. Gregorio and A. Orsatti, Uniqueness and existence of dualities over compact rings, *Tsukuba J. Math.* **18** (1994), 39–61.
14. D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274** (1982), 399–443.
15. P. J. Hilton and U. Stammbach, "A Course in Homological Algebra," 2nd ed., Springer-Verlag, New York, 1997.
16. S. MacLane, "Categories for the Working Mathematician," Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York, 1971.
17. C. Menini, Gabriel–Popescu type theorems and graded modules, *in* "Perspectives in Ring Theory (Antwerp, 1987)," pp. 239–251, Kluwer Academic, Dordrecht, 1988.
18. C. Menini and A. Orsatti, Representable equivalences between categories of modules and applications, *Rend. Sem. Math. Univ. Padova* **82** (1989), 203–231.
19. G. Mezzetti, Topological Morita equivalences induced by ideals generated by dense idempotents, *J. Algebra* **201** (1998), 167–188.
20. Y. Miyashita, Tilting modules of finite projective dimension, *Math. Z.* **193** (1986), 113–146.
21. A. Orsatti, "Una introduzione alla teoria dei moduli," Aracne, Rome, 1996 (in Italian).
22. N. Popescu, "Abelian Categories with Applications to Rings and Modules," London Mathematical Society Monographs, No. 3, Academic Press, London, 1973.
23. J. J. Rotman, "An Introduction to Homological Algebra," Academic Press, New York, 1979.
24. B. Stenström, "Rings of Quotients," Grundlehren der mathematischen Wissenschaften, Vol. 217, Springer-Verlag, Berlin/New York, 1975.
25. J. Trlifaj, Every $*$-module is finitely generated, *J. Algebra* **169** (1994), 392–398.
26. R. Wisbauer, "Foundations of Module and Ring Theory," Gordon & Breach, Reading, 1991.